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Complex Hyperspherical Equations, Nodal-Partitioning, and First-Excited-State Theorems in $\mathbb{R}^n$

Richard L. Liboff[1]

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Three problems related to the spherical quantum billiard in $\mathbb{R}^n$ are considered. In the first, a compact form of the hyperspherical equations leads to their complex contracted representation. Employing these contracted equations, a proof is given of Courant's nodal-symmetry intersection theorem for "diagonal eigenstates" of "spherical-like" quantum billiards in $\mathbb{R}^n$. The second topic addresses the "first-excited-state theorem" for the spherical quantum billiard in $\mathbb{R}^n$. Wavefunctions for this system are given by the product form, $(1/\rho^q)Z_{q+\ell}(\rho)Y_\ell^{(n)}$, where ρ is dimensionless displacement, ℓ is angular-momentum number, q is an integer function of dimension, $Z(\rho)$ is either a spherical Bessel function (n odd) or a Bessel function of the first kind (n even) and θ represents $(n-1)$ independent angular components. Generalized spherical harmonics are written $Y_\ell^{(n)}(\theta)$. It is found that the first excited state (i.e., the second eigenstate of the Laplacian) for the spherical quantum billiard in $\mathbb{R}^n$ is n-fold degenerate and a first excited state for this quantum billiard exists which contains a nodal bisecting hypersurface of mirror symmetry. These findings establish the first-excited-state theorem for the spherical quantum billiard in $\mathbb{R}^n$. In a third study, an expression is derived for the dimension of the ℓth irreducible representation ("irrep") of the rotation group $O(n)$ in $\mathbb{R}^n$ by enumerating independent degenerate product eigenstates of the Laplacian.

KEY WORDS: Hyperspherical equations; complex representation; first exited-state theorem; spherical quantum billiard; irreducible representations.

1. INTRODUCTION

Many components of the spherical quantum billiard in $\mathbb{R}^n$ (also called the "Infinite Hyperspherical Well") have been examined. Application to physics has been examined mainly with respect to many-body theory (Avery, 1989; Ballot and Navarro, 1975; Clark and Green, 1980; Cooper and Kouri, 1972; de la Ripella, 1993; de la Ripelle, 1983; Ermolaev and Sochilin, 1964; Fung, 1977; Knitk, 1974; Smith, 1960). The present work addresses three related topics. In the first of these,

[1] Schools of Electrical Engineering and Applied Physics, and Center for Applied Math, Cornell University, Ithaca, New York, 14853; e-mail: rll9@cornell.edu.

a contracted form of the hyperspherical equations is presented that gives rise to a complex representation of these equations. These relations yield necessary conditions for Courant's noda-intersection symmetry theorem (Courant and Hilbert, 1953) in $\mathbb{R}^n$. The second topic addresses extension of the first-excited theorem (Allessandrini, 1994; Amar *et al.*, 1991; Jerison, 1991; Letz, 1975; Liboff, 1994a,b,c, 2001; Lin, 1987; Melas, 1992; Sommerfeld, 1958) for regular convex polyhedra quantum billiards in $\mathbb{R}^n$ to the spherical quantum billiard in $\mathbb{R}^n$. This theorem states that for any regular convex polyhedron quantum billiard in $\mathbb{R}^n$, a first excited state (i.e., second eigenstates of the Laplacian) exists whose nodal surface is a bisecting surface of mirror symmetry and that the degeneracy of this state is the dimension of the billiard.

Wavefunctions in hyperspherical coordinates are given by products of radial functions and generalized spherical harmonics, $Y_\ell^{(n)}(\theta)$. Radial functions are solutions of a generalized radial equation and are of the form, $(1/\rho^q)Z_{q+\ell}(\rho)$, where q is an integer function of dimension, n, angular-momentum number is ℓ, and ρ is dimensionless displacement. For odd n, $Z(\rho)$ is a spherical Bessel function and for n even, $Z(\rho)$ is a Bessel function of the first kind. The odd solution reduces to standard form in $\mathbb{R}^3$. The corresponding radial equation generates all order spherical Bessel functions and Bessel functions of the first kind. It is found that the first excited state for the spherical quantum billiard in $\mathbb{R}^n$ is n-fold degenerate and that a first excited state exists for this quantum billiard that contains a nodal bisecting surface of mirror symmetry. These results are in accord with the first-excited-state theorem stated above. In a closely related problem, a sequential inequality of Bessel-function zeros is presented which implies that a second excited state (third eigenstate of the Laplacian) of the spherical quantum billiard in $\mathbb{R}^n$ exists whose nodal surfaces are likewise composed of hyperplanes of mirror symmetry. Degeneracy of this second eigenstate is $(n-1)(n+2)/2$. It is shown that $Y_\ell^{(n)}(\theta)$ is equal to a sum of products, each of degree ℓ, of hyperspherical coordinates. Lastly, an expression is derived for the dimension of the ℓth irreducible representation ("irrep") of the rotation group $O(n)$ in $\mathbb{R}^n$. [Here we are discussing the $O(n)$ group excluding inversions, usually written $O(n)^+$.] Basis functions for the ℓth irrep of $O(n)$ are given by the eigenfunctions for the hyperspherical quantum billiard in $\mathbb{R}^n$.

2. SPHERCAL QUANTUM BILLIARD IN $\mathbb{R}^n$

We consider the spherical quantum billiard in $\mathbb{R}^n$, bounded by the spherical surface S^{n-1}. Previous studies of S^n have addressed graphics (Kocak and Laidlaw, 1987), the classical mechanics of two uncoupled harmonic oscillators (Kocak *et al.*, 1988), two-body correlations and scattering amplitudes, respectively, in many-dimensional space (de la Ripella, 1993; de la Ripelle, 1983), and many-electron atoms (Avery, 1989; Knitk, 1974).

2.1. Hyperspherical Equations; $S(n, q)$ Functions

The angular description of S^{n-1} is given in terms of hyperspherical coordinates $\{\theta_i, n \geq 2\}$. Equations relating these to cartesian coordinates $\{x_k, (1 \leq k \leq n)\}$ involve the angular forms:

$$
\begin{aligned}
S(n, n-k) &\equiv \sin\theta_n \, \sin\theta_{n-1} \cdots \sin\theta_{n-k} \\
S(n, n) &= \sin\theta_n; \; S(n, 0) \equiv 1
\end{aligned}
\tag{1a}
$$

These hyperspherical equations are given by

$$
\begin{aligned}
x_n &= S(n, 0) \cos\theta_n \\
x_{n-1} &= S(n, n) \cos\theta_{n-1} \\
x_{n-2} &= S(n, n-1) \cos\theta_{n-2} \\
x_{n-3} &= S(n, n-1) \sin\theta_{n-2} \\
x_{n-4} &= S(n, n-2) \cos\theta_{n-4} \\
x_{n-5} &= S(n, n-2) \sin\theta_{n-4} \\
x_{n-6} &= S(n, n-3) \cos\theta_{n-6} \\
x_{n-7} &= S(n, n-3) \sin\theta_{n-6} \\
&\vdots \\
x_1 &= S(n, 3) \sin\theta_2 = S(n, 2)
\end{aligned}
\tag{1b}
$$

Note that x_n, x_{n-1}, and x_1 individually maintain their forms with change in n. In the preceding, the angles θ_i are defined for $i \geq 2$.

2.2. Generalized Polar and Azimuthal Angles

To identify polar and azimuthal angles in the transformation equations (1a), we consider first the angular description on S^2, namely ($n = 3$)

$$
\begin{aligned}
x_3 &= \cos\,\theta_3 \\
x_2 &= \sin\theta_3 \, \cos\theta_2 \\
x_1 &= \sin\theta_3 \, \sin\theta_2
\end{aligned}
\tag{2a}
$$

These relations identify θ_3 as the "polar" angle and θ_2 as the "azimuthal angle." The angular description on S^3 is given by ($n = 4$)

$$
\begin{aligned}
x_4 &= \cos\theta_4 \\
x_3 &= \sin\theta_4 \, \cos\theta_3
\end{aligned}
$$

$$x_2 = \sin\theta_4 \, \sin\theta_3 \, \cos\theta_2 \tag{2b}$$
$$x_1 = \sin\theta_4 \, \sin\theta_3 \, \sin\theta_2$$

Again, θ_4 may be identified as the "polar" angle and θ_2, θ_3 as azimuthal angles. Generalizing to the angular description of S^{n-1}, we see that θ_n is the generalized polar angle and θ_i, $2 \le i \le n-1$ are generalized azimuthal angles.

The transformation (1) satisfies the relation

$$\sum_{i=1}^{n} x_i^2 = 1 \tag{2c}$$

$$-1 \le x_i \le 1 \tag{2d}$$

The relation (2c) is a result of the orthogonality of the elements $\{x_i\}$ and describes the unit sphere in $\mathbb{R}^n$. With (2c) we write,

$$\sum_{i=1}^{n-1} x_i^2 = 1 - x_n^2 = \sin^2\theta_n \tag{2e}$$

At any value of x_n in the domain (2d), the relation (2e) describes a hypersphere in $\mathbb{R}^{n-1}$ of radius, $\sin\theta_n$. The hypersurface $x_n = 0$ is a bisecting hypersurface of mirror symmetry of S^{n-1}.

2.3. Complex Representation

For n even, (1b) comprises a complete set of $n/2$ couplets. Accordingly, we define the complex variable,

$$z_q \equiv x_q + i x_{q-1} \tag{3a}$$

Thus

$$\begin{aligned} z_{n-2} &= S(n, n-1) \exp(i\theta_{n-2}) \\ z_{n-4} &= S(n, n-4) \exp(i\theta_{n-4}) \\ &\vdots \\ z_4 &= S(n, 4) \exp(i\theta_4) \\ z_2 &= S(n, 3) \exp(i\theta_2) \end{aligned} \tag{3b}$$

In these relations, for odd n, effect the changes:

$$\exp(i\theta_k) \to i \exp(-i\theta_k), \quad [2 \le k \le n-2] \tag{3c}$$

and for $(n \ge 3)$,

$$(n-2, n-4, \cdots) \to (n-1, n-3, \cdots)$$

The terms, x_n, x_{n-1} may be written in the complex representation

$$\begin{aligned} z_n &= \exp i\Psi \\ \tan\Psi &= \tan\theta_n \, \cos\theta_{n-1} \end{aligned} \tag{3d}$$

Incorporating z_n into (3b) gives $[n/2, (n+1)/2]$ equations for n (even, odd).

The relations (3) play a role in the development of generalized spherical harmonics in the expansion (6b) and Courant's nodal symmetry partitioning theorem (Section 4). As an example of the mapping (3), consider the case $n = 5$. The variables x_5, x_4 are given by (1b) and x_3, x_2, x_1 by

$$x_3 + ix_2 = z_3 = iS(5,4)\,\exp(-i\theta_3) = \sin\theta_5\,\sin\theta_4(\cos\theta_3 + i\sin\theta_3)$$

$$x_1 = \mathrm{Re}\, z_1 = \mathrm{Re}\,[iS(5,1)\,\exp(-i\theta_1)] = \sin\theta_5\,\sin\theta_4\,\sin\theta_3\,\sin\theta_2$$

2.4. Radial Dependence

We turn next to the radial dependence of solutions of the Helmholtz equation

$$\triangle\Psi + k^2\Psi = 0 \tag{4a}$$

in a spherical domain bounded by S^{n-1}, on which

$$\Psi(S^{n-1}) = 0 \tag{4b}$$

In (4a), Δ represents the Laplacian in $\mathbb{R}^n$ and k^2 represents an eigenenergy. The Laplacian operator has the spherical-coordinate representation (Taylor, 1986)

$$\Delta = \hat{O}_r + r^{-2}\Delta_S \tag{5a}$$

$$\hat{O} \equiv -[\partial^2/\partial r^2 + (n-1)r^{-1}\partial/\partial r] \tag{5b}$$

Where Δ_S represents the Laplacian on the unit spherical surface, S^{n-1}, and r is written for the magnitude of the radial vector in $\mathbb{R}^n$. In (5b) a sign change was introduced consistent with a quantum mechanical representation of the kinetic energy operator (Liboff, 1998). The eigenvalue equation for the Laplacian on the unit sphere, S^{n-1}, is given by (Avery, 1989; Gallot *et al.*, 1990)

$$\Delta_S Y_\ell^{(n)} = \ell(\ell + n - 2)Y_\ell^{(n)} \tag{6a}$$

where $Y_\ell^{(n)}$ is the restriction to S^{n-1} of a homogeneous polynomial of degree ℓ in $\mathbb{R}^{n-1}$, and is given by the multinomial expansion

$$Y_\ell^{(n)} = A\sum_{\{qi\}} \frac{\ell!}{q_1!q_2!\ldots q_n!}(x_1)^{q_1}(x_2)^{q_2}\cdots(x_n)^{q_n} \tag{6b}$$

$$q_1 + q_2 + \cdots + q_n = \ell \tag{6c}$$

$$q_i \geq 0,\ \ell \geq 0,\ n \geq 2, \text{ and are integers}$$

The summation in (6b) is over all nonnegative integers subject to the conditions (2c) and (6c). The coefficient A is a normalization constant. The function $Y_\ell^{(n)}$ is constant if and only if, $\ell = 0$. In general, $Y_\ell^{(n)}$ is equal to a sum of products, each of degree ℓ, of hyperspherical coordinates. The degeneracy, $g(n, \ell)$, of a state of given ℓ in $\mathbb{R}^{n-1}$ is equal to the number of independent terms in (6b). As shown in Section 4, $g(n, \ell)$ is given by

$$g(n, \ell) = B'(n, \ell) = \delta(n, \ell) \tag{7a}$$

where $B'(n, \ell)$ is a displaced binomial coefficient and $\delta(n, \ell)$ represents dependent product states, for which $\delta(n, 2) = 0$, for all integer $n > 2$.

Apart from factors of $\hbar^2$ (Planck's constant), (6a) is an eigenvalue equation for the square of generalized angular momentum, L^2, in $\mathbb{R}^n$. Thus, with $(\Delta_S \to L^2)$, in $\mathbb{R}^3$, (6a) returns the well known expression

$$L^2 Y_\ell^{(3)} = \hbar^2 \ell(\ell + 1) Y_\ell^{(3)} \tag{7b}$$

We introduce the product solution to (4a)

$$\Psi_\ell^{(n)}(r, \theta) = w(r) Y_\ell^{(n)} \tag{8a}$$

where θ represents $(n - 1)$ independent variables. The $Y_\ell^{(n)}$ functions are labeled generalized spherical harmonics. Substituting (8a) into (4a) gives

$$\frac{\hat{O}_r w}{w} - \frac{\ell(\ell + n - 2)}{r^2} + k^2 = 0 \tag{8b}$$

or, more explicitly

$$r^2 w'' + (n - 1) r w' + [k^2 r^2 - \ell(\ell + n - 2)] w = 0 \tag{8c}$$

where primes denote differentiation with respect to r. Introducing $\rho \equiv kr$ gives the "generalized radial equation"

$$\rho^2 w'' + (2 + b)\rho w' + [k^2 \rho^2 - \ell(\ell + 1 - b)] w = 0 \tag{9a}$$

$$b \equiv n - 3 \tag{9b}$$

In $\mathbb{R}^3$, $b = 0$ and (9a) reduces to the well-known Bessel and Neumann functions (Jackson, 1999). (In this derivation, as in the case for Bessel functions in $\mathbb{R}^3$, one ignores the series that does not satisfy inversion symmetry.) A power series solution of (9a) that starts as ρ^λ reveals the indicial equation

$$\lambda(\lambda + b) = \ell(\ell + 1 + b) \tag{10a}$$

which has the two solutions

$$\lambda = \ell \tag{10b}$$

$$\lambda = -\ell - (1 + b) \tag{10c}$$

The ρ^{ℓ} behavior may be identified with spherical Bessel functions and the $\rho^{-\ell-(1+b)}$ behavior with spherical Neumann functions. To investigate orthogonality of these functions we revert to (8b) whose differential operator is rendered self-adjoint by the factor ρ^b. One obtains the following orthogonality relation for the spherical quantum billiard on the unit sphere in $\mathbb{R}^n$.

$$\int_0^1 dr\, r^{2+b} w(k_1 r) w(k_2 r) = 0 \tag{10d}$$

where k_1 and k_2 are respective zeros of w-functions. It is noted that (10d) reduces to standard orthogonality conditions for Bessel functions of the first kind in $\mathbb{R}^3$. In the following, based on boundedness of the wavefunction at the origin, Neumann functions are deleted in solutions. The general solution of (9a) is given by (Kamke, 1967)

$$w(\rho) = \frac{1}{\rho^{Q/2}} Z_\nu(\rho) \tag{11a}$$

$$Q \equiv 1 + b = n - 2 \tag{11b}$$

$$\nu^2 = \frac{1}{4}[Q^2 + 4\ell(\ell + Q)] = \frac{1}{4}(Q + 2\ell)^2 \tag{11c}$$

where $Z_\nu(\rho)$ represents Bessel functions of order ν. Consider first the case that dimension, n, is even, so that $Q = 2q$, where $q \geq 0$, and integer. In this event, the solution (11a) becomes

$$w^{(\mathrm{e})}(\rho) = \frac{1}{\rho^q} J_{q+\ell}(\rho) \tag{12a}$$

where we have written

$$\nu^{(\mathrm{e})} = (q + \ell) \tag{12b}$$

which is an integer and $J^{a+\ell}(\rho)$ is a Bessel function of the first kind (Watson, 1966). In $\mathbb{R}^2$, $Q(n = 2) = 0$, and $q = 0$. In this event, $w^{(\mathrm{e})}(\rho)$ is relevant to the unit circle quantum billiard (or circular membrane) (Liboff, 1999). For odd n, we write, $Q = 2q + 1$, where, again, $q \geq 0$, and is integer. There results,

$$w^{(\mathrm{o})}(\rho) = \frac{1}{\rho^{(2^q+11)/2}} Z_{\nu(\mathrm{o})}(\rho) \tag{13a}$$

where

$$\nu^{(\mathrm{o})} = \frac{1}{2} + (\ell + q) \geq \frac{1}{2} \tag{13b}$$

is a half odd integer and $w^{(\mathrm{o})}(\rho)$ is a spherical Bessel function (Jackson, 1999).

That is, for n odd (apart from a factor of π)

$$w^{(\mathrm{o})}(\rho) = \frac{1}{\rho^q} j_{q+\ell}(\rho) \tag{13c}$$

In $\mathbb{R}^3$, $Q(n = 3) = 1$, and $q = 0$. In this case $w^{(0)}(\rho)$ reduces to a standard Bessel function. Note that

$$q^{(\mathrm{e})} = (n-2)/2, \qquad q^{(\mathrm{o})} = (n-3)/2, \quad n \geq 2 \tag{13d}$$

In this manner we find that solutions to (9a) can be classified according to whether the dimensionalilty n is even or odd. For n even, these solutions are weighted Bessel functions of the first kind, whereas for odd n they are weighted spherical Bessel functions. In either case, the order of the Bessel function that enters is $\nu = q + \ell \geq 0$, for $n \geq 2$. Eigenfunctions for the spherical quantum billiard problem in $\mathbb{R}^n$ are given by

$$\Psi_\ell^{(n)}(r, \theta) = w_{q+\ell}(kr) Y_\ell^{(n)}(\theta) \tag{14}$$

where $w_{q+l}(kr)$ is written for either Bessel functions of the first kind (12) for n even, or spherical Bessel functions (13) for n odd and (q, ℓ) are positive integers. Eigenenergies are given by $(k_{\nu,j})$ (Smith, 1960) where $k_{\nu,j}$ is the Jth zero of w_ν. That is, $w_{\nu(\ell,n)}(k_{\nu,j}) = 0$, corresponding to Dirichlet boundary conditions on the unit sphere.

2.5. Angular Dependence

We turn next to the angular component solutions $Y_\ell^{(\mathrm{n})}(\theta)$ and properties of the ground and first-excited states of the spherical quantum billiard in $\mathbb{R}^n$. With reference to (6), we note that the eigenstate of lowest angular eigenvalue is

$$Y_0^{(n)}(\theta) \equiv K_n = \text{ constant} \tag{15a}$$

corresponding to the radial component, $W_q(kr)$. The first-degree harmonic polynomial in $\mathbb{R}^n$ is given by

$$Y_1^{(n)} = A_1^{(n)}(x_1 + x_2 + \cdots + x_n) \tag{15b}$$

where $A_1^{(n)}$ is constant. The preceding polynomial corresponds to the radial component $w_{q+1}(kr)$ in (13). The second-degree harmonic polynomial in $\mathbb{R}^n$ is given by [see (6b)]

$$Y_2^{(n)} = A_2^{(n)}\left(x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n + x_1^2 + \cdots x_{n-1}^2\right) \tag{15c}$$

There are

$$\binom{n}{2} + (n-1) = \frac{(n-1)(n+2)}{2} \tag{15d}$$

independent terms in the sum (15c). The polynomial $Y_2^{(n)}$ matches with the $w_{q+2}(kr)$ radial component in (14).

As a specific example of the angular wavefunction, $Y_1^{(n)}$, consider the state

$$Y_1^{(n)} = A_1^{(n)} x_n = A_1^{(n)} \cos\theta_n \tag{15e}$$

which vanishes on the bisecting hypersurface of S_{n-1}, $\theta_n = \pi/2$. With this result and the form (6b) we note that any $Y_\ell^{(n)}$ function has a component that vanishes on a bisecting hypersurface of S_{n-1}.

3. WAVEFUNCTIONS

The ground state of the spherical quantum billiard in $\mathbb{R}^n$, for $n \geq 2$, is

$$\Psi_0^{(n)}(r, \theta) = K_n w_{q+0}(k_{0,1} r) \tag{16}$$

with corresponding eigenenergy, $(k_{0,1})$ (Smith, 1960). The eigenfunction (16) vanishes on S^{n-1} and is otherwise nonnodal in the unit sphere in $\mathbb{R}^n$. Note in particular, that this ground state includes zero value of angular-momentum ℓ-value in all $\mathbb{R}^n$. Recalling that Bessel functions are interlaced and that the first finite zero (Watson, 1966) of $Z_\nu(\rho)$ grows with ν, it follows that a first-excited-state for the spherical quantum billiard in $\mathbb{R}^n$ is given by

$$\Psi_1^{(n)}(r, \theta) = w_{q+1}(k_{1,1} r) Y_1^{(n)}(\theta) \tag{17}$$

with corresponding eigenenergy $(k_{1,1})$ (Smith, 1960). One component of the preceding solution includes the factor Y_1 which, as noted above, vanishes on a bisecting hypersurface of S^{n-1}. Furthermore, as follows from (15d), the state (17) is n-fold degenerate. The nodal structure of the first excited state (17), as well as the nonnodal property of the ground state (16) are both in accord with Courant's nodal partitioning theorem (Courant and Hilbert, 1953). These properties establish the first-excited-state theorem for the spherical quantum billiard in $\mathbb{R}^n$.

3.1. Second Excited State

We consider the second excited state of the spherical quantum billiard in $\mathbb{R}^n$ (i.e., the third eigenstate of the Laplacian). It is noted that zeros of Bessel functions satisfy the following inequality sequence.

$$0 < k_{\nu,1} < k_{\nu+1,1} < k_{\nu+2,1} < k_{\nu,2} \tag{18}$$

where, as above, $k_{\nu,s}$ is the sth zero of Z_ν. Validity of the first three left inequalities of (18) is evident. The last inequality on the right of (18) follows from Porter's theorem (Watson, 1966) which states that an odd number of $k_{\nu+2}$ zeros exist between successive k_ν zeros. With the preceding sequence, it follows that the angular component of the second excited state is given by (15c) corresponding to $\ell = 2$. Choosing a particular component of the series (15c) we write

$$Y_2^{(n)}(r,\theta) = A_2^{(n)} \cos\theta_n \sin\theta_n \cos\theta_{n-1} \tag{19}$$

which is zero on the hypersurfaces, $\theta_n = \pi/2$, $\theta_{n-1} = \pi$ and $\theta_n = \pi$, $\theta_{n-1} = \pi/2$, corresponding to the mutually perpendicular hyperplanes, $x_n = 0$, $x_{n-1} = 0$. These hyperplanes partition the unit sphere in $\mathbb{R}^n$ into four subdomains of equal measure. Again, this result is consistent with Courant's nodal partitioning theorem (Courant and Hilbert, 1953). With (19), a second eigenstate of the spherical quantum billiard in $\mathbb{R}^n$ is given by

$$\Psi_2^{(n)}(r,\theta) = w_{q+2}(k_{2,1}r)Y_2^{(n)}(\theta) \tag{20a}$$

As follows from (6c), degeneracy of the second excited state of the spherical quantum billiard in $\mathbb{R}^n$ is

$$g(n,2) = (n+2)(n-1)/2 \tag{20b}$$

4. NODAL INTERSECTION SYMMETRY THEOREM

A theorem described by Courant (Courant and Hilbert, 1953) states that if a set of nodals of a solution to the Helmholtz equation in a convex domain in $\mathbb{R}^2$, with a smooth boundary, intersect, then these nodals make an equal-angled array about the point of intersection. Note that this is the case for the circular harmonics (Liboff, 1994a,b,c), as well as for the eigenfunction (19). With the present formalism, this theorem may be generalized to $\mathbb{R}^n$. Nodal surfaces of a given eigenstate in $\mathbb{R}^n$ are obtained by setting the right side of (6b) equal to zero and equating each respective independent sum of terms to zero. We define a "spherical-like" billiard (Liboff, 2002) in $\mathbb{R}^n$ as a convex billiard with a smooth boundary whose ratio of minimum-to-maximum diameters, x, is such that $1 > x > 0$, $1 - x << 1$. Consider that a number of nodal surfaces of a given eigenstate of this billiard intersect at a point. Mapping this point onto the origin of $\mathbb{R}^n$ indicates that the solution in an infinitesimal neighborhood about this point is given by (14) with $Y_\ell^{(n)}$ given by (6b). The eigenstates contained in (6b) include a subset of "diagonal" states that are sums over single-coordinate states of the form x_ν^q. Expressing these states in terms of the complex representation (3b) gives, in accord with (6b),

$$Y_\ell^{(n)}(diag) = z_2^{q_1} + z_4^{q_1} + \cdots \tag{21a}$$

To insure independence of the terms in this sum, we introduce the following procedure: In the first step, the last term in the sequence (21a) is eliminated. The remaining terms contain all x_i except x_n and x_{n-1} and are independent. In the following step, the penultimate term in the sequence (21a) is eliminated. The remaining terms contain all x_i variables except x_{n-3} and x_{n-4} and are independent.

In each step of this procedure, the resulting sequence in (21a) contains independent terms and vanishes providing each individual term in the respective sum vanishes. Consider, for example, $Y_k^{(4)}$ in the complex representation (3):

$$Y_k^{(4)}(\theta_2, \theta_3, \theta_4) = (z_2)^k = (\sin\theta_3 \ \sin\theta_4)^k \ \exp(ik\theta_2) \tag{21b}$$

Nodals of both real and imaginary parts of this relation partition the (x_3, x_4) hyperplane into $2k$ wedges of equal angle, respectively [see (1d)]. Continuing in this manner we find that nodals of the state (21a) partition all hyperplanes (x_q, x_{q+1}) into $2k$ wedges of equal angle, respectively. We may conclude that these diagonal eigenstates of the Helmholtz equation in $\mathbb{R}^n$ for a spherical-like quantum billiard, satisfy Courant's nodal-symmetry partitioning theorem.

5. DIMENSIONS OF IRREPS OF THE SPHERICAL QUANTUM BILLIARD IN $\mathbb{R}^n$; YOUNG SYMMETRIZERS

The group-theoretic technique to determine the dimensions of irreps of $O(n)$ generates an algorithm for these entities. It is based on Young symmetrizes (Hamermesh, 1962) to a subspace of nonvanishing traceless tensors of non-negative integer rank for a given value of n, [derived from the general linear group, $GL(n)$] to obtain traceless tensors of a given symmetry type (i.e., symmetric, antisymmetric and mixed) with respect to tensor-index sequences. Sets of tensors of a given symmetry comprise a basis of an irrep of $O(n)$. However, the related algorithm does not generate a closed expression for the dimensions of the irreps of $O(n)$.

5.1. Degeneracies and Basis Functions

In the present work, an alternative procedure is described to obtain the dimensions of the irreps of $O(n)$, based on the following: If the symmetries of an Hamiltonian, H, are described by the group, G, then the dimensions of irreps of G are equal to the degeneracies of of the eigenstates of H. The degeneracy, $g(n, \ell)$, of the (n, ℓ) eigenstate of the spherical quantum billiard in $\mathbb{R}^n$ is equal to the number of independent terms on the right side of (6b) subject to the constraint (6c). Thus, calculating the degeneracy, $g(n, \ell)$, of the (n, ℓ) eigenstate is related to the number of independent ordered sequences in the expression (Riordan, 1958)

$$\sum_{\{q_i\}} x_1^{q_1} x_2^{q_2} x_3^{q_3} \cdots x_n^{q_n} \tag{22a}$$

As each ordered sequence is at the same (n, ℓ) values, degeneracy of this (n, ℓ) state is equal to the number of partitions of ℓ into n slots with related q_i-labels, and is given by the displaced binomial coefficient

$$g(n, \ell) = B'(n, \ell) \equiv B(n, \ell) - 1 + 2\delta_{2,n}(1 - \ell) \tag{22b}$$

$$B(n, \ell) = \binom{n + \ell - 1}{\ell} \tag{22c}$$

The kronecker-delta symbol, $\delta_{2,n}$, serves to insure the value, $g(2, \ell) = 2$. For the $(n, 2)$ state, (22b) gives $(n > 2)$

$$g(n, 2) = \binom{n + 1}{2} - 1 = \frac{(n - 1)(n + 2)}{2} \tag{23a}$$

in accord with (15d). More generally, $g(n, \ell)$ may be less than that given by (22b) due to dependence of product states in (22a). To account for such states, we write

$$g(n, \ell) = B'(n, \ell) - \delta(n, \ell) \tag{23b}$$

From the known fact that $g(3, \ell) = 2\ell + 1$, one finds

$$\delta(3, \ell) = \frac{\ell^2 - \ell - 2}{2} \tag{23c}$$

whereas with (22b) one obtains $\delta(n, 2) = 0$.

The function $g(n, \ell)$ gives the dimensions of irreps of the rotation group $O(n)$ in $\mathbb{R}^n$. As the Laplacian in $\mathbb{R}^n$ is invariant to rotations about the origin, it follows that eigenfunctions (14) of this operator are a basis of the ℓth irrep of $O(n)$. We note that there is a countably infinite number of irreps of $O(n)$.

6. CONCLUSIONS

A contracted form of the hyperspherical equations was employed in formulating a complex representation of these hyperspherical equations. With this complex representation, it was shown that "diagonal" eigenstates of the Helmholtz equation in $\mathbb{R}^n$ for a spherical-like convex quantum billiard, satisfy Courant's nodal symmetry partitioning theorem. Working in hyperspherical coordinates, it was found that wavefunctions are given by products of Bessel functions and generalized spherical harmonics. The Bessel functions are solutions of a generalized radial equation. Solutions separate according to whether dimension number, n, is even or odd. For even n, solutions of the radial equation are weighted Bessel functions of the first kind and for n odd, are weighted spherical Bessel functions. It was found that the first excited state for the spherical quantum billiard in $\mathbb{R}^n$ is n-fold degenerate and that a first excited state for this quantum billiard exists which contains a nodal bisecting hypersurface of mirror symmetry. These properties establish the first-excited-state theorem for this system. A sequential inequality of Bessel-function zeros was noted to imply that a second excited state exists of the spherical quantum billiard in $\mathbb{R}^n$ whose nodal surface is likewise composed of hyperplanes of mirror symmetry. It was noted further that the angular harmonic, $Y_\ell^{(n)}(\theta)$ is equal to a sum of products, each of degree ℓ, of hyperspherical coordinates. Explicit forms of $Y_\ell^{(1)}\theta$, and $Y_\ell^{(2)}\theta$ were given. An expression was obtained for the dimension of the ℓth irrep of the $O(n)$ group in $\mathbb{R}^n$ by counting independent degenerate eigenstates of the Laplacian. It was noted that basis functions for the ℓth irrep of the $O(n)$ group are given by derived $\Psi_\ell^{(n)}$ eigenfunctions for the spherical quantum billiard in $\mathbb{R}^n$.

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